A NOTE ON PI-RINGS

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ABSTRACT

The classical result that an algebra which satisfies a polynomial identity satisfies a power $S_{2n}[x]^m = 0$ of the standard identity, is generalized to arbitrary rings.

Let R be an algebra over a commutative ring Ω and assume that R satisfies a polynomial identity $p[x_1, \cdots, x_k] = \sum_{r,(i)} \alpha_{(i)} x_{i_1} \cdots x_{i_r}$ of degree d (not necessarily homogeneous). Let $\Omega_0 = \{\alpha_{(i)}\}$ be the set of all coefficients of the polynomial $p[x]$.

THEOREM. *If R satisfies $p[x] = 0$, then R satisfies also all the identities $\Omega_0^m S_{2n}[x_1, \cdots, x_{2n}]^m = 0$ for some integer m, and $n = [d/2]$, and where $S_{2n}[x] = \sum \pm x_{j_1} \cdots x_{j_{2n}}$ is the standard identity of degree $2n$.*

PROOF. Let $X = \{(r_1, \cdots, r_{2n});\ r_i \in R\}$ be the set of all $2n$-tuples of elements of R, and consider the ring R^X of all functions from X into R. Clearly R^X is also an Ω-algebra and being a product of rings R, it also satisfies the identity $p[x] = 0$.

Let N be the lower radical of R^X, then N is an Ω-ideal and so R^X/N is a semi-prime ring satisfying the identity $p[x] = 0$. If $R^X/N = 0$, then $R^X = N$ is nil. In particular, the function f_1 defined by $f_1(r_1, \cdots, r_{2n}) = r_1$ is nil, and so it follows that $f_1^m = 0$ and hence $r_1^m = f_1^m(r_1, \cdots, r_{2n}) = 0$, i.e., R is a nil ring of bounded index. This clearly implies that R satisfies our theorem.

If $R^X/N \neq 0$, then this ring satisfies theorem 12 of [1], and by translating the proof of this theorem to R^X, we conclude that R^X contains two ideals M_1, M_2 such that:

(i) $M_1 = \cap \{P;\ P$ a prime ideal satisfying $\Omega_0 R \not\subseteq P\} = \{f \in R^X;\ \Omega_0 f \subseteq N\}$.

(ii) $M_2 = \cap \{P;\ P$ a prime ideal such that $\Omega_0 R \subseteq P\}$.

(iii) M_1, M_2 are Ω-ideals and $M_1 \cap M_2 = N$.

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(iv) $\Omega_0 R \subseteq M_2$ and $R/M_1 \hookrightarrow M_n(K)$ where K is a commutative Ω-algebra and, therefore, R/M_1 satisfies the standard identity $S_{2n}[x] = 0$.

This implies that for arbitrary $f_i \in R^X$ we have $\Omega_0 S_{2n}[f_1, \cdots, f_{2n}] \subseteq \Omega_0 M_1 \cap \Omega_0 R \subseteq M_1 \cap M_2 = N$. Finally, N is a locally nilpotent nil ring and since the set $\Omega_0 S_{2n}[f_1, \cdots, f_{2n}] = \{\alpha_{(i)} S_{2n}[f_1, \cdots, f_{2n}];\ \alpha_{(i)} \in \Omega_0\}$ is finite, it follows that there exists an integer m such that $(\Omega_0 S_{2n}[f_1, \cdots, f_{2n}])^m = 0$. In particular, we could have taken f_i to be the function $f_i(r_1, \cdots, r_n) = r_i$ which will imply that

$$0 = \Omega_0^m S_{2n}[f_1, \cdots, f_{2n}]^m (r_1, \cdots, r_n) = \Omega_0^m S_{2n}^m [r_1, r_2, \cdots, r_{2n}].$$

Q.E.D.

An important corollary:

COROLLARY. *If $p[x] = 0$ is a regular identity of R, i.e., "$\Omega_0 r = 0 \Rightarrow r = 0$", then R satisfies $S_{2n}^m[x] = 0$ (which is, in particular, an identity with integral coefficients and one of them is* 1).

Indeed, if $\Omega_0^m S_{2n}^m [r_1, \cdots, r_{2n}] = 0$, then by the regularity of $p[x] = 0$ it follows that $S_{2n}^m[x] = 0$ holds in R.

REFERENCE

1. S. A. Amitsur, *Prime rings having polynomial identities with arbitrary coefficients*, Proc. Lond. Math. Soc. **17** (1967), 470–486.

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